EXERCISE – V HINTS & SOLUTIONS

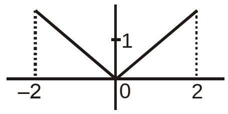
Sol.1 A

$f(x) = x$ for $0 < x \le 2$

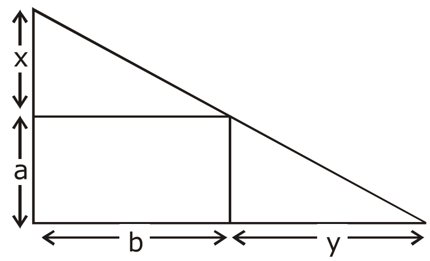
$= -x$ for $-2 \le x < 0$

$= 1$ for $x = 0$

Local maxima at $x = 0$



Sol.2 A,B



Area of triangle $A = \frac{1}{2}(a + x)(y + b)$

$$\frac{x}{b} = \frac{a}{y} \Rightarrow ab = xy$$

$$A = \frac{1}{2}(ay + ab + xy + bx)$$

$$A = \frac{1}{2}\left[\frac{a^2b}{x} + 2ab + bx\right] = \frac{b}{2}\left[\frac{a^2}{x} + 2a + x\right]$$

$$\frac{dA}{dx} = \frac{b}{2}\left[-\frac{a^2}{x^2} + 1\right]$$

for max. & min. value of A $\frac{dA}{dx} = 0$

$x^2 = a^2 \Rightarrow x = \pm a \Rightarrow x = a$

$$\frac{d^2A}{dx^2} = \frac{b}{2}\left[\frac{2a^2}{x^3}\right]$$

$$\left.\frac{d^2A}{dx^2}\right|_{x=a} = \frac{a^2b}{a^3} = \frac{b}{a} > 0$$

A is minimum

min. Area $A = \frac{1}{2}(a + a)(b + b) = 2ab$

Sol.3 (a) D

$f(x) = (1 + b^2)x^2 + 2bx + 1$

$f'(x) = 2(1 + b^2)x + 2b = 0$

$$\Rightarrow x = \frac{-b}{(1+b^2)}$$

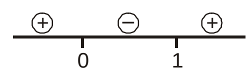
$$m(b) = (1 + b^2)\frac{b^2}{(1+b^2)^2} - \frac{2b^2}{(1+b^2)} + 1$$

$$= \frac{b^2}{(1+b^2)} - \frac{2b^2}{(1+b^2)} + 1$$

$$m(b) = \frac{1}{(1+b^2)} = y$$

$$1 + b^2 = \frac{1}{y} \qquad \Rightarrow b^2 = \frac{1}{y} - 1$$

$$b^2 = -\frac{(y-1)}{y} \ge 0 \Rightarrow \frac{y-1}{y} \le 0$$



$y \in (0, 1]$

Range $\in (0, 1]$

(b) A

$\cot\alpha_1 . \cot\alpha_2$ .... $\cot\alpha_n = 1$

possible when $\alpha_1 = \alpha_2 = \alpha_3 = ... = \alpha_n = \frac{\pi}{4}$

Max. value $P = (\cos\alpha_1) . (\cos\alpha a_2)$ .... $(\cos\alpha_n)$

$$= \left(\frac{1}{\sqrt{2}}\right)^n = \frac{1}{2^{n/2}}$$

Sol.4 A

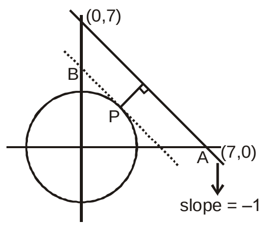
AM $\ge$ GM

$$\frac{(a_1 + a_2 + ... + a_{n-1} + 2a_n)}{n} \ge (a_1a_2 \ ...... \ 2a_n)^{1/n}$$

$$\frac{a_1 + a_2 + ... + 2a_n}{n} \ge (2e)^{1/n}$$

$$a_1 + a_2 + ... + a_{n-1} + 2a_n \ge n(2e)^{1/n}$$

Sol.5 **(a)** $\frac{x^2}{6} + \frac{y^2}{3} = 1$



Let point

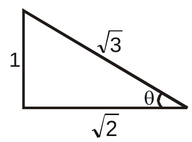
P($\sqrt{6}\cos\theta, \sqrt{3}\sin\theta$)

$\frac{2x}{6} + \frac{2y}{3}\, y' = 0$

$y' = \left.\frac{x}{2y}\right|_p = -\frac{1}{2}\frac{\sqrt{6}}{\sqrt{3}}\frac{\cos\theta}{\sin\theta}$

$y' = -\frac{1}{\sqrt{2}}\cot\theta = -1$

$\tan\theta = \frac{1}{\sqrt{2}}$



$\cos\theta = \frac{\sqrt{2}}{\sqrt{3}}$; $\sin\theta = \frac{1}{\sqrt{3}}$

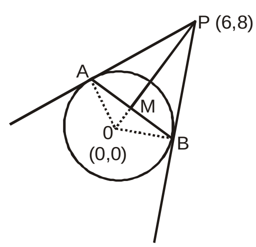
P(2, 1)

(b) $x^2 + y^2 = r^2$

Line : AB $\quad 6x + 8y = r^2 \quad$ .....(1)

$PM = \left|\frac{36 + 64 - r^2}{10}\right|$

$= \left|\frac{100 - r^2}{10}\right|$



OB = r

$OM = \left|\frac{r^2}{10}\right|$; $BM = \sqrt{r^2 - \frac{r^4}{100}} = r\sqrt{1 - \frac{r^2}{100}}$

Area = $\frac{1}{2}$ PM × AB

$= \frac{1}{2}\left(\frac{100 - r^2}{10}\right) 2r\sqrt{1 - \frac{r^2}{100}}$

$A = r\frac{(100 - r^2)^{3/2}}{10}$

$\frac{dA}{dr} = \frac{3}{2}\frac{(100 - r^2)^{1/2}}{10}(-2r^2) + \frac{1}{10}(100 - r^2)^{3/2} = 0$

$-3r^2 + (100 - r^2) = 0$

$4r^2 = 100$

$\Rightarrow r = 5$

Sol.6 **(a)** $f(x) = x^3 + bx^2 + cx + d \;,\; 0 < b^2 < c$

$f'(x) = 3x^2 + 2bx + c$

$D = 4b^2 - 12c = 4(b^2 - c) - 8c$

$\Rightarrow D < 0$

$f'(x) > 0$

$\Rightarrow$ f(x) is increasing in $x \in (-\infty, \infty)$

(b) Let $f(x) = 3x^2 + (3 - 2\pi)x - \pi \sin x$

$f'(x) = 6x + 3 - 2\pi - \pi\cos x$

$f''(x) = 6 + \pi\sin x > 0$ for $x \in \left[0, \frac{\pi}{2}\right]$

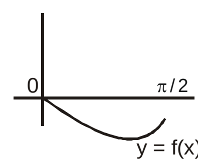
$f'(x)$ is increasing in $\left[0, \frac{\pi}{2}\right]$

$f(0) = 0$

$f\left(\frac{\pi}{2}\right) < 0$

$\Rightarrow$ graph of f(x) always lies below the x-axis

$f(x) \le 0$ is $x \in \left[0, \frac{\pi}{2}\right]$



$3x^2 + 3x \le 2\pi x + \pi\sin x$

$\Rightarrow \sin x + 2x \ge \frac{3x(x+1)}{\pi}$

Sol.7 Let the polynomial $P(x) = ax^3 + bx^2 + cx + d$

Given that

$P(-1) = -a + b - c + d = 10$

$P(1) = a + b + c + d = -6$

$P'(1) = 3a - 2b + c = 0$

$P''(1) = 6a + 2b = 0 \Rightarrow 3a + b = 0$

solving for a, b, c, d we get

$f(x) = x^3 - 3x^2 - 9x + 5$

$P'(x) = 3x^2 - 6x - 9 = 0 \quad \Rightarrow x = -1$ and $x = 3$

x = –1 is the max. and x = 3 min.

Hence point (–1, 10) and (3, –22)

Distance = $\sqrt{(-1-3)^2 + (10+22)^2} = 4\sqrt{65}$

Sol.8 **(a)** Let $f(x) = ax^3 + bx^2 + cx + d$

$f(2) = 8a + 4b + 2c + d = 18$

$f(1) = a + b + c + d = -1$

$f'(-1) = 3a - 2b + c = 0$

$f''(0) = 0 \Rightarrow b = 0$

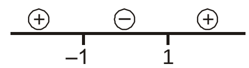
By solving we get the polynomial

$$f(x) = \frac{1}{4}(19x^3 - 57x + 34)$$

$$f'(x) = \frac{1}{4}[57x^2 - 57] = 0$$

$\Rightarrow x = \pm 1$

min. at $x = 1$



increasing for $x \in (1, 2\sqrt{5}\,]$

(b) $$g'(x) = f(x) = \begin{cases} e^x & , \ 0 \le x \le 1 \\ 2 - e^{x-1} & , \ 1 < x \le 2 \\ x - e & , \ 2 < x \le 3 \end{cases}$$

$g'(x) = 0$ when $x = 1 + \ell n\, 2$ and $x = e$

$$g''(x) = \begin{cases} -e^{x-1} & 1 < x \le 2 \\ 1 & 2 < x \le 3 \end{cases}$$

$g''(1 + \ell n\, 2) = -e^{\ell n\, 2} < 0$

$\Rightarrow x = 1 + \ell n\, 2$ is a max. point

$g''(0) = 1 > 0$ $\quad$ $x = e$, is a min. point.

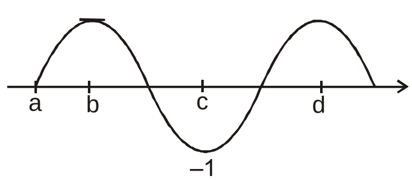
(c) Let $h(x) = f(x)\, f'(x)$

from this graph f(x) is zero at atleast four places

and f '(x) is zero at atleast three places

hence h(x) is zero in atleast 7 places

graph of f(x)

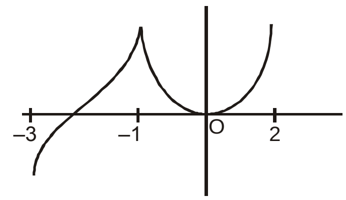


hence h'(x) is zero in atleast 6 places.

$\therefore$ $h'(x) = g(x) = 0$ has minimum 6 solutions.

Sol.9 **(a)** $f(x) = (2 + x)^3 \quad -3 < x \le -1$

$\quad\quad = x^{2/3} \quad -1 < x < 2$



clearly

$x = 0$ is point of minima

and $x = -1$ is point of maxima

(b) $$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}$$

$$f(x) = 1 - \frac{2ax}{x^2 + ax + 1}$$

$$f'(x) = -\left[\frac{(x^2 + ax + 1)2a - 2ax(2x + a)}{(x^2 + ax + 1)^2}\right]$$

$$= \frac{2a(x^2 - 1)}{(x^2 + ax + 1)^2}$$

(i) $$f''(x) = 2a\left[\frac{(x^2 + ax + 1)^2(2x) - 2(x^2 - 1)(x^2 + ax + 1)(2x + a)}{(x^2 + ax + 1)^4}\right]$$

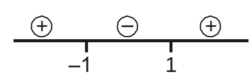
$$= 2a\left[\frac{2x(x^2 + ax + 1) - 2(x^2 - 1)(2x + a)}{(x^2 + ax + 1)^3}\right]$$

$$f''(1) = \frac{4a(a + 2)}{(a + 2)^3} = \frac{4a}{(a + 2)^2}$$

$$f''(-1) = 2a\left[\frac{(-2)\,(2 - a)}{(2 - a)3}\right] = \frac{-4a}{(a - 2)^2}$$

$(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 4a - 4a = 0$

(ii)



$\downarrow (-1, 1)$ and min. at $x = 1$

(iii) $$g(x) = \int_0^{e^x} \frac{f'(t)}{1 + t^2}\, dt$$

$$g'(x) = \frac{f'(e^x)}{1 + e^{2x}}\, e^x$$

$$= \frac{2a\,(e^{2x}-1)}{(e^{2x}+ae^{x}+1)^2} \times \frac{e^x}{1+e^{2x}}$$

$g'(x) = 0$; if $e^{2x} - 1 = 0$ i.e. $x = 0$

If $\quad x < 0, \quad e^{2x} < 1 \;\Rightarrow\; g'(x) < 0$

Sol.10 **(a)** Let $p(x) = ax^4 + bx^3 + cx^2 + dx + e$

$p'(x) = 4ax^3 + 3bx^2 + 2x + d$

$p'(1) = 4a + 3b + 2c + d = 0 \quad .....(1)$

$p'(2) = 32a + 12b + 4c + d = 0 \quad .....(2)$

$$\lim_{x\to 0}\left(1+\frac{p(x)}{x^2}\right) = 2$$

$$\lim_{x\to 0}\left(\frac{ax^4+bx^3+(c+1)x^2+dx+e}{x^2}\right) = 2$$

$c + 1 = 2, \qquad d = 0, e = 0$

$c = 1$

from (1) and (2)

$4a + 3b = -2 \quad$ and $\quad 32a + 12b = -4$

$a = 1/4, b = -1$

$$p(x) = \frac{x^4}{4} - x^3 + x^2$$

$$p(2) = \frac{16}{4} - 8 + 4 = 0$$

(b) $\quad f(x) = 2x^3 - 15x^2 + 36x - 48$

$A = \{x/x^2 + 20 \le 9x\}$

$= \{x/x \in [4, 5]\}$

$f'(x) = 6(x^2 - 5x + 6) = 0$

$x = 2,3$

$f(2) = -20,\; f(3) = -21,\; f(4) = -16,\; f(5) = 7$

maximum value $= f(5) = 7$

Sol.11 (a) D

$f'(x) > 0,\; g'(x) > 0,\; h'(x) > 0$. Maximum will occur at 1 $\quad$ so $f(1) = g(1) = h(1)$

(b)

$g'(x) / g(x) = 2010\,(x - 2009)\,(x - 2010)^2$..........

.... $(x - 2011)^3\,(x - 2012)^4$

For maximum, $g'(x)$ must change its sign from positive to negative which is true only at $x = 2009$.

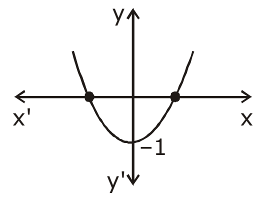
Sol.12 Let $f(x) = x^4 - 4x^3 + 12x^2 + x - 1$

$f'(x) = 4x^3 - 12x^2 + 24x+1$

$f''(x) = 12x^2 - 24x+24$

$f''(x) = 12(x - 1)^2 + 12>0$

$\Rightarrow f'(x) \uparrow$



$f(0) = -1$

slope of $f(x)$ is increasing

Hence there are 2 distinct real roots.

Sol.13 $p'(x) = a(x - 1)(x - 3) = ax^2 - 4ax + 3a$

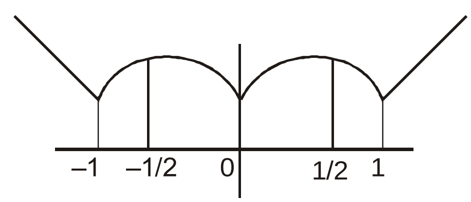
$$p(x) = \frac{ax^3}{3} - 2ax^2 + 3ax + c$$

$$\frac{a}{3} - 2a+3a + c = 6 \;\;\&\;\; 9a - 18a+9a = 2$$

$\Rightarrow a = 3$

$p'(0) = 3a = 9$

Sol.14
$$f(x) = \begin{cases} x^2+x-1 & ;\; x = 1 \\ -x^2+x+1 & ;\; 0 \le x \le 1 \\ -x^2-x+1 & ;\; -1 \le x \le 0 \\ x^2-x-1 & ;\; x \le -1 \end{cases}$$



Sol15. ABCD

$$f'(x) = e^{x^2}(x-2).(x-3)$$

Increasing in $(0,2) \cup (3,\infty)$,

Decreasing in $(2,3)$

Maximum at 2 & minimum at 3

$$\& \; f''(x) = e^{x^2}(2x-3) + (x^2-5x+6)\cdot 2x\cdot e^{x^2} = 0$$

$$\Rightarrow 2x - 5 + 2x^3 - 10x^2 + 12x = 0$$

$$\Rightarrow 2x^3 - 10x^2 + 14x - 5 = 0$$

so there exists some $c \in (0,\infty)$ such that $f''(c) = 0$